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DYNAMICS OF A TSUNAMI APPROACHING THE SHORE

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ABSTRACT: The problem addressed is the propagation of a long wave onto a beach of small slope. For a tsunami, this stage of motion follows its generation and deep-water travel and precedes its propagation onto dry land. The outcomes of the study are relationships between the offshore wave height and velocity, and the waterlevel and velocity structure of the wave as it runs onto the beach.

The Eulerian mass and momentum conservation equations have been used. As in the method of characteristics, physical variables are transformed as linear combinations of the independent variables (position and time) and the dependent variables (velocity and waterlevel). With these transformations the equations are replaced by a pair of quasi-linear first-order partial differential equations. The latter equations are then formally solved as a pair of integral equations. For simple initial conditions, such as a Gaussian wave offshore in still water, the integral equations may be directly evaluated. The solution is in terms of the transformed variables and it is not obvious that the inverse transformation is always possible. Limitations of the method are discussed.

1. INTRODUCTION

Design of structures and settlements to survive tsunami requires the ability to predict the depths of inundation, wave runup and the water velocities within a tsunami. The motion of a tsunami onto the shore may be divided into four stages as follows: generation, travel across the ocean, propagation onto the shore and propagation over dry land and inland. The design data are obtained from the last two stages. The last two stages and some relevant scales are discussed in Hinwood (2005). This paper addresses the problem of the propagation of a tsunami onto the shore – the second last stage - but not across dry land.

In the following section, formal solutions are obtained for the velocities and waterlevels within the wave as it runs onto a beach. Detailed solutions are not presented here but the penultimate section discusses some of the difficulties of obtaining solutions in terms of the basic independent variables (position and time).

2. ANALYSIS

.The quasilinear unsteady flow up a plane beach of small slope $\tan\alpha \approx \alpha$, as shown in Fig. 1 is described by the solutions to the equations expressing the flux balances of mass and momentum, respectively, taken over a vertical section and written in the form

$$\frac{\partial}{\partial t} h(x,t) + \frac{\partial}{\partial x}\{u(x,t)[x\tan\alpha + h(x,t)]\} = 0 \qquad (1)$$

$$\frac{\partial}{\partial t} u(x,t) + u(x,t)\frac{\partial}{\partial x}u(x,t) + g\frac{\partial}{\partial x}h(x,t) = f(u(x,t)). \qquad (2)$$

Here $u \equiv u\ (x, y)$ is the horizontal flow velocity assumed uniformly constant over the vertical section, $h \equiv (x, y)$ is the flow displacement in the vertical direction relative to the undisturbed position of the water surface and g is the acceleration due to gravity. The horizontal coordinate x is measured out from O, the undisturbed beach point and t is the time variable.

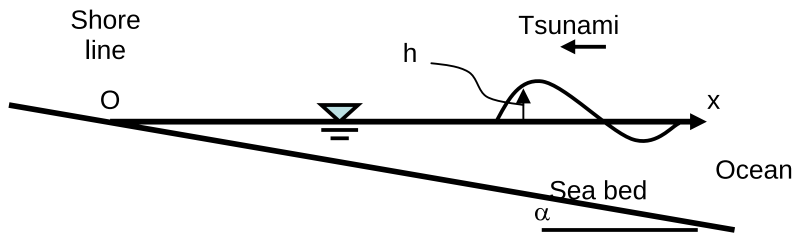

Fig. 1 - Tsunami approaching a plane beach.

Note that in equations (1) and (2) the pressure is assumed hydrostatic and measured by the height of the displaced water surface above its undisturbed value. On the right hand side of (2) the term $f(u(x,t))$ is intended to represent the resistance to flow due to the presence of the sea bed and will influence both the speed and height of an incoming wave. However, as we shall see, the simple analysis given here, which is a reworking of that given by Carrier et al, (2003), cannot be carried through when this drag term is present, although estimates of the drag interpreted as a momentum flux can be constructed outside the flow equations and after determining the flow field. In any case, equations (1), (2) may not be valid at the beach point, except in a very general sense, since vertical flows may be induced here. However, these matters aside, we shall proceed on the assumption that the solutions to the flow equations without the drag term will give a not unreasonable description of a tsunami flow up a sloping beach.

The first step is to set the flow equation in dimensionless form. An appropriate length scale is some length L referring to a point in the undisturbed sea surface well away from the beach where an initial incoming wave might have negligible effect at the undisturbed beach point. Using this value we have the scales

$$\text{length} = L, \quad \text{time} = \sqrt{L/\alpha g}, \quad \text{speed} = \sqrt{\alpha L g}, \quad \text{surface displacement} = \alpha L \tag{3}$$

Using these scales the flow equations, neglecting the drag term, become

$$\frac{\partial}{\partial t} h(x,t) + \frac{\partial}{\partial x}\{u(x,t)\,[x + h(x,t)]\} = 0 \tag{4}$$

$$\frac{\partial}{\partial t} u(x,t) + \frac{\partial}{\partial x}\left\{h(x,t) + \frac{1}{2}u(x,t)^2\right\} = 0. \tag{5}$$

These nonlinear first order partial differential equations form a hyperbolic pair of equations with real characteristics in the (x, t) domain, as in the case of simple flow in a rectangular channel. $u(x, t)$ and $h(x, t)$ with given initial values all values of x and with boundary conditions such that u and h are finite everywhere over all values of x for all $t > 0$. The equations can be solved numerically, using either finite differences or the method of characteristics although, as we shall see, it is possible to construct analytical solutions. An advantage of proceeding numerically is that drag due to the sea bed can be taken into account which is not simply the case in deriving the analytical solutions.

To construct the analytical solution to the flow equations (4), (5) we must introduce a transformation that will allow the conversions of these two non linear equations into a pair of linear first order partial differential equations, again of hyperbolic type, a property that must be maintained. Now, as with flow in a channel, we find in the present case that the characteristic directions in the x, t plane satisfy

$$\left.\frac{dx}{dt}\right|^{\pm} = u \pm \sqrt{x + h(x,t)}. \tag{6}$$

The expression $x + h(x, t)$ is part of the definition of the characteristic lines, and hence of the transformation that we are seeking. We need another similar variable involving the variable t in a similar manner. Noting that the flow is to be defined in the domain $x > 0$, $t > 0$ we rewrite (6) in the form

$$\left.\frac{d(x-t)}{dt}\right|^{\pm} = -(t-u) \pm \sqrt{x + h(x,t)}. \tag{7}$$

Now set

$$\xi = x + h(x,t), \quad \eta = t - u(x,t) \tag{8}$$

and (7) becomes

$$\left.\frac{d(x - t)}{dt}\right|^{\pm} = -\eta \pm \sqrt{\xi} \tag{9}$$

That is, the characteristic directions are defined in terms of the 'natural' variables (ξ, η) with reference to the line $x - t = 0$. If we set $\delta = +\sqrt{\xi}$ then we can say that the region of influence about the point ($x - t$, t) is contained in the wedge ($-\eta \pm \delta$). The value $\eta = 0$ can be considered as referring to a disturbance coming into the beach while the value $\delta = 0$ refers to the 'run-up' on the beach.

We shall now use the variables ξ, η to effect a coordinate transformation $(x, t) \rightarrow (\xi, \eta)$ which essentially interchanges the roles of the dependent and independent variables and which converts the nonlinear equations into a pair of linear first order partial differential equations, as first described by Tuck and Hwang (1972). Proceeding formally, we define the transformation

$$\xi = x + h(x,t), \quad \eta = t - u(x,t). \tag{10}$$

Once the functions h(x, t) and u (x, t) are known these equations define the variables x and t implicitly in terms of the variable ξ and η. In order to avoid confusion when using the variables ξ and η we shall write

$$\mathfrak{h}(\xi,\eta) = h(x(\xi,\eta), t(\xi,\eta)), \quad \mathfrak{u}(\xi,\eta) = u(x(\xi,\eta), t(\xi,\eta)). \tag{11}$$

We now apply the transformation (10) to the flow equations to obtain linear equations with derivatives with respect to ξ and η rather than nonlinear equations where the derivatives are with respect to x and t. To effect this transformation consider first

$$\begin{aligned} d\xi &= \frac{\partial \xi}{\partial x} dx + \frac{\partial \xi}{\partial t} dt \\ &= \frac{\partial \xi}{\partial x} \left\{ \frac{\partial x}{\partial \xi} d\xi + \frac{\partial x}{\partial \eta} d\eta \right\} + \frac{\partial \xi}{\partial t} \left\{ \frac{\partial t}{\partial \xi} d\xi + \frac{\partial t}{\partial \eta} d\eta \right\} \end{aligned}$$

from which we get two equations for $\dfrac{\partial \xi}{\partial x}$ and $\dfrac{\partial \xi}{\partial t}$:

$$\begin{aligned} \frac{\partial \xi}{\partial x}\frac{\partial x}{\partial \xi} + \frac{\partial \xi}{\partial t}\frac{\partial t}{\partial \xi} &= 1 \\ \frac{\partial \xi}{\partial x}\frac{\partial x}{\partial \eta} + \frac{\partial \xi}{\partial t}\frac{\partial t}{\partial \eta} &= 0. \end{aligned} \tag{12}$$

Further from $d\eta = \dfrac{\partial \eta}{\partial x} dx + \dfrac{\partial \eta}{\partial t} dt$ we get two equations for $\dfrac{\partial \eta}{\partial x}$ and $\dfrac{\partial \eta}{\partial t}$:

$$\begin{aligned} \frac{\partial \eta}{\partial x}\frac{\partial x}{\partial \xi} + \frac{\partial \eta}{\partial t}\frac{\partial t}{\partial \xi} &= 0 \\ \frac{\partial \eta}{\partial x}\frac{\partial x}{\partial \eta} + \frac{\partial \eta}{\partial t}\frac{\partial t}{\partial \eta} &= 1. \end{aligned} \tag{13}$$

Now from the definitions of ξ and η in terms of x and t we have

$$\frac{\partial x}{\partial \xi} = 1 - \frac{\partial h}{\partial \xi}, \quad \frac{\partial x}{\partial \eta} = -\frac{\partial h}{\partial \eta}, \quad \frac{\partial t}{\partial \xi} = \frac{\partial u}{\partial \xi}, \quad \frac{\partial t}{\partial \eta} = 1 + \frac{\partial u}{\partial \eta}.$$

Using these in (12) and (13) and solving we get

$$
\begin{aligned}
J\frac{\partial \xi}{\partial x} &= \frac{\partial t}{\partial \eta} = 1 + \frac{\partial u}{\partial \eta} \\
J\frac{\partial \xi}{\partial t} &= -\frac{\partial x}{\partial \eta} = \frac{\partial h}{\partial \eta} \\
J\frac{\partial \eta}{\partial x} &= -\frac{\partial t}{\partial \xi} = -\frac{\partial u}{\partial \xi} \\
J\frac{\partial \eta}{\partial t} &= \frac{\partial x}{\partial \xi} = 1 - \frac{\partial h}{\partial \xi}
\end{aligned}
$$

where the Jacobian, J, is given by

$$
J = \frac{\partial x}{\partial \xi}\frac{\partial t}{\partial \eta} - \frac{\partial x}{\partial \eta}\frac{\partial t}{\partial \xi} \tag{14}
$$

The transformation $x,\ t \rightarrow \xi,\ \eta$ is assumed everywhere nonsingular so that J nowhere vanishes. Similarly, the inverse transformation is assumed everywhere nonsingular so that $1/J$ nowhere vanishes.

Using the above results we have for the derivatives with respect to the variables x and t :

$$
\begin{aligned}
J\frac{\partial}{\partial x} &= J\left\{\frac{\partial \xi}{\partial x}\frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial x}\frac{\partial}{\partial \eta}\right\} \\
&= \left(1 + \frac{\partial u}{\partial \eta}\right)\frac{\partial}{\partial \xi} - \frac{\partial u}{\partial \xi}\frac{\partial}{\partial \eta}
\end{aligned} \tag{15}
$$

$$
\begin{aligned}
J\frac{\partial}{\partial t} &= J\left\{\frac{\partial \xi}{\partial t}\frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial t}\frac{\partial}{\partial \eta}\right\} \\
&= \frac{\partial h}{\partial \eta}\frac{\partial}{\partial \xi} + \left(1 - \frac{\partial h}{\partial \xi}\right)\frac{\partial}{\partial \eta}
\end{aligned} \tag{16}
$$

Using these results in the flow equations (4), (5) we have for the resulting equations

$$
\frac{\partial}{\partial \xi}(\xi\,\mathfrak{u}) + \frac{\partial}{\partial \eta}\left(\mathfrak{h} + \mathfrak{u}^2/2\right) = 0 \tag{17}
$$

$$
\frac{\partial \mathfrak{u}}{\partial \eta} + \frac{\partial}{\partial \xi}\left(\mathfrak{h} + \mathfrak{u}^2/2\right) = 0. \tag{18}
$$

where the old English font has been used for h($\xi,\ \eta$), u($\xi,\ \eta$) in contrast to $h(x,\ t)$, $u(x,\ t)$.

Setting E = h + u^2/2 and eliminating E between (17) and (18) we get

$$
\frac{\partial^2}{\partial \xi^2}(\xi\,\mathfrak{u}) - \frac{\partial^2 \mathfrak{u}}{\partial \eta^2} = 0 \tag{19}
$$

while eliminating u we get

$$
\frac{\partial}{\partial \xi}\left(\xi\frac{\partial \mathfrak{E}}{\partial \xi}\right) - \frac{\partial^2 \mathfrak{E}}{\partial \eta^2} = 0 \tag{20}
$$

Solving these two equations by separating the variables we have for the flow functions u($\xi,\ \eta$) and E($\xi,\ \eta$):

$$
\mathfrak{u}(\xi,\eta) = \frac{1}{\sqrt{\xi}}\int_{\rho=0}^{\infty}\mathbf{J}_1(2\rho\sqrt{\xi})\,\rho\,d\rho \times \{A_u(\rho)\sin(\rho\eta) + B_u(\rho)\cos(\rho\eta)\} \tag{21}
$$

$$
\mathfrak{E}(\xi,\eta) = \int_{\rho=0}^{\infty}\mathbf{J}_0(2\rho\sqrt{\xi})\,\rho\,d\rho \times \{A_{\mathfrak{E}}(\rho)\sin(\rho\eta) + B_{\mathfrak{E}}(\rho)\cos(\rho\eta)\} \tag{22}
$$

where J0 and J1 are the zero order and first order Bessel functions of the first kind. The four functions $A_u(\rho)$, $B_u(\rho)$, $A_E(\rho)$, $B_E(\rho)$ are not all independent since the solutions (21) and (22) must satisfy the two original flow equations (17) and (18). Substituting we find $A_u(\rho) = -B_E$ and $B_u(\rho) = A_E$. so that finally

$$\mathfrak{u}(\xi,\eta) = \frac{1}{\sqrt{\xi}} \int_{\rho=0}^{\infty} \mathbf{J}_1(2\rho\sqrt{\xi})\,\rho\, d\rho \times \{A(\rho)\cos(\rho\eta) - B(\rho)\sin(\rho\eta)\}, \quad (23)$$

$$\mathfrak{h}(\xi,\eta) + u(\xi,\eta)^2/2 = \int_{\rho=0}^{\infty} \mathbf{J}_0(2\rho\sqrt{\xi})\rho\, d\rho \times \{A(\rho)\sin(\rho\eta) + B(\rho)\cos(\rho\eta)\}, \quad (24)$$

on using the definition

$$\mathfrak{E}(\xi,\eta) = \mathfrak{h}(\xi,\eta) + \mathfrak{u}(\xi,\eta)^2/2 .$$

The functions $A(\rho)$ and $B(\rho)$ are to be determined for each initial value problem.

3. APPLICATION

As an indication of the method of solution, consider the class of problems where the flow field is static at $t = 0$. That is, at $\eta = 0$, with $u(x, 0) = 0$ then $\eta = t - u(x, t) = 0$. From this it follows that $A(\rho) = 0$ and the above representations (23) and (24) reduce to the forms

$$\mathfrak{u}(\xi,\eta) = -\frac{1}{\sqrt{\xi}} \int_{\rho=0}^{\infty} \mathbf{J}_1(2\rho\sqrt{\xi})\,B(\rho)\sin(\rho\eta)\,\rho\, d\rho, \quad (25)$$

$$\mathfrak{h}(\xi,\eta) + \mathfrak{u}(\xi,\eta)^2/2 = \int_{\rho=0}^{\infty} \mathbf{J}_0(2\rho\sqrt{\xi})\, B(\rho)\cos(\rho\eta)\,\rho\, d\rho. \quad (26)$$

The function $B(\rho)$ is to be determined, in the present instance, from the form of $h(\xi, 0)$. Thus, suppose we are given a known form $h_0(\xi) = h(\xi, 0)$ which we set in (26) with $\eta = 0$:

$$\mathfrak{h}_0(\xi) = \int_{\rho=0}^{\infty} \mathbf{J}_0(2\rho\sqrt{\xi})\,B(\rho)\,\rho\, d\rho. \quad (27)$$

To solve this integral equation for $B(\rho)$ it is convenient to transform the variable ξ such that $2\sqrt{\xi} = \sigma^2$. The source function $h_0(\xi)$ will become $k_0(\sigma)$, say, and we have

$$\mathfrak{k}_0(\sigma) = \int_{\rho=0}^{\infty} \mathbf{J}_0(\rho\sigma)\,B(\rho)\,\rho\, d\rho,$$

so that $k_0(\sigma)$ is the zero order Hankel transform of $B(\rho)$. Inverting this transform we get

$$B(\rho) = \int_{\sigma=0}^{\infty} \mathbf{J}_0(\rho\sigma)\,\mathfrak{k}_0(\sigma)\,\sigma\, d\sigma. \quad (28)$$

Now introduce the kernel function $K_u(\sigma, \eta; \zeta)$ defined as

$$\mathcal{K}_u(\sigma,\eta;\zeta) = (2/\sigma)\int_{\rho=0}^{\infty} \mathbf{J}_0(\rho\zeta)\,\mathbf{J}_1(\rho\sigma)\sin(\rho\eta)\,\rho\, d\rho. \quad (29)$$

Writing $v(\sigma, \eta) = u(\xi, \eta)$ we have

$$\mathfrak{v}(\sigma,\eta) = -\int_{\zeta=0}^{\infty} \mathcal{K}_u(\sigma,\eta;\zeta)\,\mathfrak{k}_0(\sigma)\,\zeta\, d\zeta. \quad (30)$$

In a similar way we have the kernel function $K_E(\sigma, \eta; \zeta)$.

For initial conditions which are sufficiently simple to permit these integrals to be evaluated analytically, analytical solutions may be obtained. There are several classes of initial condition which are amenable to integration, the most obvious being an initial displacement of the water surface, $\eta(x,0)$, in an otherwise quiescent sea, e.g. Carrier et al (1958, 2003). Guard et al (2005) have shown that waves produced using the condition of no initial velocity produce waves in the near-shore zone which are much lower than the observed waves arising from a given $\eta(x,0)$.

In general the integral equations must be evaluated numerically. There is a considerable advantage in delaying the use of numerical solution to this stage as against numerically solving the primitive long wave equations. The possibility of numerical instability is avoided, accuracy is more readily controlled, and the computational time is greatly reduced. However, there are difficulties in moving from the world of the 'characteristic' variables ξ and η to the real world of x and t.

Consider the initial value problem where the water surface is locally (and quiescently) disturbed some distance (L) from the beach point O, as, for example, by the square wave displacement

$$h(x,0) = \begin{cases} h_{00}\,, & a < x < b\,, \\ 0\,, & \text{otherwise} \end{cases} \qquad (31)$$

where h_{00} is a constant. Referring to equation (8) it may be seen that the values of ξ over the interval $a < x < b$ will be the same as those outside this interval, over a length segment determined by the values of h_{00} and α. The inverse transformation will then be indeterminate. Outside the region traversed by a positive wave, the inverse will not be indeterminate and hence the leading edge of a positive wave may be found. The problem may be overcome to some extent in the case that the integral equations are evaluated numerically by following points on the wave as the wave propagates.

4. VALIDITY OF THE LONG WAVE SOLUTIONS

4.1 General considerations

The validity of the solutions of this flow, as a model of the run up of a tsunami, may be assessed in the following terms:

(i) Factors included in the equations
(ii) Factors omitted from the equations
(iii) Simplifications made to obtain the solutions.

The non-linear long wave equations include the forces due to hydrostatic pressure, and local and advective accelerations. Thus they can describe a transient event and one which changes as it propagates. These capabilities are required to analyse a wave running onto a beach. However, solutions to these equations show that the wave generated by a wide range of initial conditions will break, raising concerns about the validity of the solutions which are addressed below.

The long wave equations omit friction and do not include the effects of curvature of the water surface (which causes departures from hydrostatic pressure). These two omissions are discussed below.

For analytical simplicity we have considered, as have most others, a wave running normal to the contours onto a plane beach. This is the problem considered here and no additional simplifications have been required to obtain solutions. In applying solutions of this problem to any real problem there will be simplifications and approximations, but it does form a relevant base case and displays the main features of the wave propagation more clearly than a case with complex geometry and boundary conditions.

4.2 Wave breaking

It has been shown by Greenspan (1958) that all positive waves (water depth increasing as the wave front passes a fixed point) break at the wave front before reaching the undisturbed shoreline, and so do steep negative waves, in direct contradiction to the finding of Carrier and Greenspan (1958). Tuck and Hwang presented computed water surface profiles and described the profiles showing "post-breaking" as "mathematical curiosities only". Carrier et al (2003) stated that for such profiles, "the interpretation becomes uncertain", but asserted that the experiments of Synolakis (1987) show "non-linear shallow-water-wave theory recovers immediately and yields extremely accurate predictions for later times".

The breaking criterion used by Tuck and Hwang, Carrier et al (1958, 2003), and most others is the condition that the water surface becomes vertical, as determined by the Jacobian, J, of the coordinate transformation becoming zero. When this occurs, the inverse transformation to (x,t) coordinates is no longer unique, implying that there is more than one depth at a given $(x.t)$ position. More directly, Greenspan (1958) used the condition that the gradient of the water surface became infinite, but developed his criterion only for the leading edge of the wave.

While we agree with these authors that a vertical tangent to the water surface may be expected to lead to the formation of a breaking wave, the collapse and runup of a surge predicted by Meyer and Taylor (1972) is another possible outcome – a finite height of vertical slope would be more certain. The disagreement over whether or not all positive waves break could be resolved with further analysis, but as discussed in the next section such effort is not warranted because of the omissions from the equations. It should be noted that most researchers into wave breaking regard the overturn or vorticity at the wave front as better criteria (e.g. Peregrine, 1983) but these measures cannot be simulated by the long wave equations.

Once the breaking criterion has been reached the authors are agreed that the solution near the breaker is not to be believed but some believe that the solution is valid over much of the (x,t) space. First we consider the following part of the wave, seaward of the "breaker". Here the linear long wave equations are applicable and show that each elemental part of a wave propagates with phase speed $c=\sqrt{g\left(h+\eta\right)}$. Near the "breaker" the fluid speed is close to this value and hence the disturbance to the flow created by the "breaker" propagates only slowly seawards. As it does so, and disturbance to the waterlevel is reduced in height by the increasing depth, $h(x)$, and thus the perturbation to the seaward depth and velocity fields is minor and diminishes seaward, vanishing asymptotically.

Considering the flow landwards of the "breaker", the "breaker may be approximated by a locally steady surge for which the phase speed is $c=\sqrt{g\left(h+\eta^{-}\right)}\sqrt{1+\left(\eta^{-}+\eta^{+}\right)/2h}$ where η^- and η^+ are the elevations seawards and landwards of the "breaker' respectively. This speed is greater than the phase speed given above and so the "breaker" runs ahead of the originating wave. Although the "breaker" is faster, the wave front is already travelling fairly close to this speed, so that the flow at any point landwards of the "breaker" is not affected until the "breaker" actually reaches that point. Hence it is to be expected that the flow will be affected by the formation of a "breaker" only in the near field, seawards of the "breaker". The differences in the flow pattern and phase speed will of course be important in the determination of wave loads on structures or seabed scour as the "breaker" passes.

4.3 Departure from hydrostatic pressure due to water surface curvature

The validity of the long wave equations is usually considered in terms of a non-linearity parameter $\varepsilon=\eta/h$ and a dispersion parameter $\sigma^2=\left(h/L\right)^2$ where L is generally taken to be the wavelength of the incoming wave (Peregrine 1972, 1983). Provided that $\varepsilon,\sigma^2<<1$ the linearised long wave equations are adequate; use of the non-linear wave equations relaxes the requirements on ε. However, these parameters increase as the wave deforms in shoaling water, the non-linearity become important first then the dispersion. Under these conditions the hydrostatic pressure assumption is no longer valid and an additional term must be added to the momentum equation, changing it to the Boussinesq equation. The additional term to be added to the right side of equation (5) is:

$$-\frac{h}{3}\frac{\partial^3\eta}{\partial x\,\partial t^2}$$

It is worth noting that although solitary waves of permanent form can exist, the long wave equations do not describe them because the wave profile deforms with the deeper elements propagating faster, while the additional (dispersion) term in the Boussinesq equations permits waves of permanent form.

The expressions obtained here should be valid provided that $\varepsilon,\sigma^2<1$. The additional term above may be evaluated following solution as a definitive check.

4.4 Significance of seabed friction

Seabed friction and internal viscous energy dissipation will reduce the wave energy propagating landwards and will generate non-uniform velocity profiles which may be expected to affect the details of wave breaking. The latter is outside the scope of the long wave analysis, but the loss of energy will be briefly considered – in a numerical solution of equations (23) and (24) the energy loss could be evaluated from the calculated velocities and depths and a more precise check performed *post hoc*.

The friction term in equation (2) may be written as $f\left(u\left(x,t\right)\right)=\mathrm{f}\,u^2/\left(h+\eta\right)$ for unidirectional flow (e.g. at the front of a positive wave), where f is the Darcy-Weisbach friction factor. Utilising the scaling in equation (3) shows that the ratio of the friction term to the accelerations is f/α; the same value is obtained for the ratio of the work done against bed friction to the initial potential energy of the wave. It is quite possible for this ratio to be of order 1 so that the neglect of friction would appear to be unwarranted.

This scaling is of a general scope, with both phase speed and particle velocities scaled in the same way. The water particle velocities are virtually negligible in deep water, where the phase speed is greatest. This fact is clearly shown in the sample velocity plots in Carrier et al (2003) which show the scaled particle velocity falls very rapidly with distance, becoming negligible at a scaled distance of 0.1 from the front and does not exceed about 0.2 anywhere. Using these numbers a more realistic estimate of the ratio of the work to the potential energy is 0.004 f/α, showing that the neglect of friction has no significance in terms of overall energy loss. Applying the Carrier et al values to the terms in equation (2), the ratio is another order of magnitude smaller for x > 0.1, but rises to about 0.04 f/α near the front of the wave.

5. CONCLUSIONS

A general solution of the equations for a tsunami approaching the shore has been given in the form of a pair of integral equations which may be evaluated to predict the tsunami waterlevels and velocities, which are the quantities required for design of structures or scour protection. The integral equations may be solved exactly for some initial conditions, e.g. simple deep-water wave forms on an initially stationary sea. More general initial conditions require the integral equations to be evaluated numerically, but this is still less demanding and more informative than solving the primitive equations numerically.

The solution is in terms of the transformed, "characteristic" coordinates. The inverse coordinate transformation to obtain the solution in terms of position and time (*x, t*) may require numerical solution and may not be single valued. The latter concern has not been resolved and may mean that only the characteristics of the front of a tsunami can be determined without further development of this method.

Wave breaking has previously been identified with a locally vertical tangent to the water surface. Although the long wave equations cannot be valid after "breaker" formation it has been shown that the "post-breaker" solution will not be affected over most of the (*x, t*) domain. While the linear equations are adequate for most of the travel of the wave, the non-linear term becomes important in the last half wavelength before the shoreline and very near "breaking" the water surface curvature results in departures from the assumed hydrostatic pressure. Thus the long wave model cannot accurately predict the location or dimensions of the broken wave.

The method does not include resistance to flow but does permit the neglected term to be evaluated to test the validity of its omission and this omission is justified from consideration of order of magnitude arguments.

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